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EDITED BY

T. A. A. BROADBENT, M.A.

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CONTENTS.

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THE OPERATION $[f(D)]^{-1}x^m e^{\lambda x}$.

BY J. P. DALTON, M.A., D.Sc., F.R.S.S.AF.

1. An interesting note in the May issue of the *Gazette* by Mr. F. Underwood * draws attention to a possible source of error in determining particular integrals by means of Forsyth's expansion theorem : †

$$[f(D)]^{-1}x^m T=\sum_{a=0}^{m} {}^mC_a x^{m-a}\left\{\frac{d^a}{dD^a}[f(D)]^{-1}\right\}T. \quad\text{.............(1)}$$

In the course of his article Mr. Underwood says : "It seems to be essential, though probably not generally known, that, in the application of this formula, the successive operators . . . must be expressed in partial fractions before being used to operate on the function T."

The object of this note is to investigate more closely, in the case of $T=e^{\lambda x}$, the apparent failure of the expansion theorem (1), and thus to show that a complete resolution of the operators into their component partial fractions is not essential, and to indicate a convenient and exact alternative procedure.

2. To facilitate printing let us write $\psi(D)$ for $[f(D)]^{-1}$, and $\psi^{(r)}(D)$ for the rth derived function of ψ. Then from (1) we have

$$\psi(D)x^m e^{\lambda x}=\sum_{a=0}^{m} {}^mC_a x^{m-a}\psi^{(a)}(D)e^{\lambda x},$$

$$=e^{\lambda x}\sum_{a=0}^{m} {}^mC_a x^{m-a}\psi^{(a)}(D+\lambda)(1). \quad\text{.............(2)}$$

We shall compare this with the result of the usual exponential operation

$$\psi(D)e^{\lambda x}x^m=e^{\lambda x}\psi(D+\lambda)x^m. \quad\text{..............................(3)}$$

3. If $f(\lambda)\neq 0$ then $\psi(D+\lambda)$ and its derived functions are expansible in a series of positive powers of D. Hence

$$\psi(D+\lambda)=\sum_{a=0}\frac{1}{a!}\psi^{(a)}(\lambda)D^{(a)}, \quad\text{..............................(4)}$$

and therefore, when the operand is unity,

$$\psi^{(\beta)}(D+\lambda)(1)=\psi^{(\beta)}(\lambda). \quad\text{..............................(5)}$$

The particular integral given by the expansion theorem (2) becomes

$$y=e^{\lambda x}\sum_{a=0}^{m} {}^mC_a x^{m-a}\psi^{(a)}(\lambda). \quad\text{..............................(6)}$$

* F. Underwood, *Math. Gaz.* xv. p. 99 (1930).

† A. R. Forsyth, *Treatise on Diff. Eqns.* 4 ed. p. 78 (1914).

From the exponential theorem (3), we also obtain

$$y=e^{\lambda x}\psi(D+\lambda)x^m$$
$$=e^{\lambda x}\sum_{a=0}^{m}\frac{1}{a!}\psi^{(a)}(\lambda)D^{(a)}x^m$$
$$=e^{\lambda x}\sum_{a=0}^{m}{}^mC_a\psi^{(a)}(\lambda)x^{m-a}. \qquad \text{.......................(7)}$$

As long, therefore, as λ is not a root of $f(D)$ there arises no difficulty in the application of (2).

4. If, on the other hand, $f(\lambda)\equiv 0$, then $\psi(D+\lambda)$ and its derived functions necessarily involve integrations as well as differentiations.

Let $$\psi(D)=(D-\lambda)^{-q}\phi(D), \qquad \text{...............................(8)}$$

where $\phi(D)$ is expansible in a series of positive powers of D. We then have

$$\psi^{(r)}(D+\lambda)(1)=\sum_{a=0}^{r}(-1)^a a!\,{}^{q+a-1}C_a\,{}^rC_a\phi^{(r-a)}(\lambda)D^{-(q+a)}(1), \qquad \text{......(9)}$$

and the expansion theorem (2) yields

$$y=e^{\lambda x}\sum_{\beta=0}^{m}\frac{x^{m+q-\beta}}{\overline{q-1}!}\phi^{(\beta)}(\lambda)\left[\sum_{a=0}^{m-\beta}(-1)^a\,{}^mC_{\beta+a}\,{}^{\beta+a}C_a\frac{1}{q+a}\right]+e^{\lambda x}P_{m+q-1} \quad \text{...(10)}$$

which readily reduces to

$$y=\frac{m!}{\overline{m+q}!}e^{\lambda x}\left[\sum_{a=0}^{m}{}^{q+m}C_a\phi^{(a)}(\lambda)x^{m+q-a}+P_{m+q-1}\right]. \qquad \text{........................(11)}$$

In (10) and (11) P_{m+q-1} is an arbitrary polynomial of degree $(m+q-1)$ in x.

Now, whenever the solution of a differential equation involves integrals of an order higher than that of the equation itself, (as, *e.g.* in simultaneous equations, or in equations of the type $y=f(x, p)$, or in the present case), the solution so obtained is of a higher order of generality than that appertaining to the given differential equation, and the proper solution must be obtained by an appropriate adjustment of the additional constants of integration. The arbitrary polynomial in (11) contains terms of degree 0 to $(m+q-1)$ in x. Of these, terms of degree 0 to $(q-1)$ are already contained in the complementary function, and may accordingly be dropped from the particular integral. The coefficients of the remaining terms are automatically evaluated in the process of solution, inasmuch as the initial group of terms in (11) are of degree q to $(m+q)$ in x. Hence the final form of the particular integral (11) is

$$y=\frac{m!}{\overline{m+q}!}e^{\lambda x}\sum_{a=0}^{m}{}^{q+m}C_a\phi^{(a)}(\lambda)x^{m+q-a}. \qquad \text{..........................(12)}$$

Application of the exponential theorem (3) gives

$$y=e^{\lambda x}\left[\sum_{a=0}^{m}{}^mC_a\frac{\overline{m-a}!}{\overline{m+q-a}!}x^{m+q-a}\phi^{(a)}(\lambda)+P_{q-1}\right], \qquad \text{.........(13)}$$

and as $e^{\lambda x}P_{q-1}$ is already contained in the complementary function this becomes

$$y=e^{\lambda x}\frac{m!}{\overline{m+q}!}\sum_{a=0}^{m}{}^{m+q}C_a x^{m+q-a}\phi^{(a)}(\lambda), \qquad \text{.........................(14)}$$

a result identical with (12).

5. In practice, if $\phi(D+\lambda)$ is a simple function, the successive values of $\phi^{(a)}(\lambda)$ may be obtained simply by straightforward differentiation. Otherwise

it may be more convenient to expand $\phi(D+\lambda)$ in a series of positive powers of D, to the required number of terms : thus

$$\phi(D+\lambda)=\sum_{a=0}^{m} A_a D^a + \ldots .$$

Then $$\phi^{(a)}(\lambda)=a!A_a, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

and the particular integral (12) becomes

$$y=\frac{m!}{\overline{m+q}!}\, e^{\lambda x} \sum_{a=0}^{m} {}^{q+m}C_a a! A_a x^{m+q-a}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

The source of error against which Mr. Underwood rightly warns us lies in using only the leading term of (16). If $f(D)$ has only the multiple root λ, then the function ϕ is a constant, and the leading term is a correct particular integral. But if $f(D)$ has at least one other root in addition to the root λ, then the leading term does not suffice. The correct particular integral could be obtained by adding to the leading term of (16) terms of degree $(m+q-1)$ to q in x, with arbitrary coefficients, and then substituting in the given differential equation ; but as a rule this method would be too laborious for practical purposes.

6. The utility of (12) may be judged from its application to two of Mr. Underwood's examples :

Example I : $$(D^2-1)y=x^m e^x ;$$
$$m=m ;\ q=1 ;\ \lambda=1.$$
$$\phi(D)=(D+1)^{-1} ;\ \phi^{(a)}(D)=(-1)^a a!(D+1)^{-(a+1)} ;\ \phi^a(1)=(-1)^a a!\,2^{-(a+1)} ;$$
$$\therefore\ y=e^x m! \sum_{a=0}^{m} (-1)^a 2^{-(a+1)} \frac{x^{m+1-a}}{\overline{m+1-a}!}.$$

Example II : $$(D^2+1)y=x\sin x=I(xe^{ix}) ;$$
$$m=1 ;\ q=1 ;\ \lambda=i.$$
$$\phi(D)=(i+D)^{-1} ;\ \phi^{(a)}(D)=(-1)^a a!(D+i)^{-(a+1)} ;\ \phi^{(a)}(i)=(-1)^a a!(2i)^{-(a+1)} ;$$
$$\therefore\ y=I[\tfrac{1}{2}e^{ix}\{(2i)^{-1}x^2-2(2i)^{-2}x\}]=-\tfrac{1}{4}(x^2\cos x - x\sin x).$$

The modification (16) is illustrated by

Example III : $$(D^2+1)(D^2+D+1)y=x^2e^{ix} ;$$
$$m=2 ;\ q=1 ;\ \lambda=i.$$
$$\phi(D+i)=-\tfrac{1}{2}\left(1-\frac{iD}{2}\right)^{-1}[1-(2-i)D-iD^2]^{-1}$$
$$=-\tfrac{1}{2}+\left(1-\frac{3i}{4}\right)D-(\tfrac{9}{8}-2i)D^2+\ldots ;$$
$$\therefore\ y=\tfrac{1}{12}e^{ix}[-2x^3+12x^2-27x+i(48x-9x^2)].$$

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June 30, 1930. J. P. Dalton.

GLEANINGS FAR AND NEAR.

804. Of geometric figures the most rare,
And perfect'st, are the circle and the square,
The city and the school much build upon
These figures, for both love proportion.
The city-cap is round, the scholar's square,
To show that government and learning are
The perfect'st limbs i' th' body of a state :
For without them, all's disproportionate.

—Thomas Dekker, *The Honest Whore*, Part the Second, Act I., Scene 3.

[Per Mr. James Buchanan.]

LIMITS IN GEOMETRY.

By Prof. E. H. Neville, M.A.*

The subject of this paper occupies a strange position among the fundamental concepts of mathematics. The beginner uses limits with equal freedom, or, as some would say, with equal recklessness, in analysis and geometry, but whereas the time comes when the analytical limit is submitted to the severest scrutiny, the examination of the geometrical limit forms no part of any university course and comes into no treatise. There is not even an essay on the subject in the collection of *Questioni riguardanti la Geometria elementare*, edited by Enriques, or a section dealing with it in Klein's *Elementarmathematik vom höhere Standpunkte aus*. When his turn comes to teach, every mathematician knows all about limits in analysis, is familiar with half a dozen ways of looking at them, and indeed has a basis of thorough understanding from which to exercise his discretion as to the treatment proper for one class of students or another. Of limits in geometry, except perhaps in three special cases, he has probably never heard a critical word.

The possible exceptions are tangents, asymptotes, and circles of curvature. Something is said of these in courses of analysis, and Fowler's Tract, *The Elementary Differential Geometry of Plane Curves* (2nd ed. 1929), handles them with the utmost precision. But always the basis is frankly analytical: to say that a variable curve Γ, dependent on parameters $u, v, \ldots$, tends to a curve Λ as $u, v, \ldots$ tend to $a, b, \ldots$, *means* that if the equation of Γ is

$$\phi(x, y; u, v, \ldots)=0,$$

then there is a function $\psi(x, y)$ such that for every fixed pair of values of x and y the function $\phi(x, y; u, v, \ldots)$ tends to $\psi(x, y)$ as $u, v, \ldots$ tend to $a, b, \ldots$, and $\psi(x, y)=0$ is the equation of Λ. It is arguable that this definition succeeds only by reducing to tautologies many of the theorems for the sake of which it is introduced, but this is not the line of thought I wish to follow up now. The question I ask is whether the definition is in any sense a clarification, a reduction to formal terms, of the vague idea in which the unsuspicious schoolboy placed instinctive confidence long before he ever heard of coordinates. I do not think it is, and what I propose to give is a geometrical theory which is related to primitive intuition but is expressible in terms adequate for exact demonstrations. For the sake of brevity I shall speak of plane geometry only, but it will be evident that the modifications for dealing with limits on any surface, or in space, are trivial.

1. We must begin with some definitions. Given any length ρ, let us say that a figure Γ is *indistinguishable* from part of a figure Δ, within the *standard of tolerance* ρ, if each point of Γ has some point or other of Δ at a distance not greater than ρ from it. As this is the fundamental definition, I want to convince the critical logician at once that no multiplicative axiom is concealed within it, to burrow and gnaw until the superstructure collapses and pretentious foundations are seen to be nothing but dry rot. For this purpose we have only to recall that if P is any point of the plane, the distances from P to the various points of Δ compose a class with a definite lower bound μ_P, dependent of course on the position of P; this lower bound, which is not necessarily attained, is well known as the distance separating P from Δ. What I am saying is that Γ is indistinguishable from part of Δ, within the standard ρ, if for every position of P in Γ the lower bound μ_P is less than or equal to ρ. Or to vary the phrasing, the distances separating the different points of Γ from Δ compose a class with a definite upper bound, and the condition of indis-

* The substance of this paper was given at a joint meeting of the Manchester Branch of the Mathematical Association and the University of Manchester Mathematical Society on February 19th, 1930, and again at the Annual Meeting of the Mathematical Association in London on January 6th, 1931.

tinguishability is that this upper bound is not greater than ρ; the upper bound may itself be infinite if Γ is unbounded, and in this case, needless to say, the condition of indistinguishability is not satisfied for any finite standard.

If each of two figures Γ, Δ is indistinguishable from part of the other, within the standard ρ, naturally we are to say simply that the figures are indistinguishable within this standard. Having modified the form of our first definition in one way for the sake of logic, let us modify the form of the second in a different way for the sake of intuition. Suppose that we have a pencil which marks every would-be point as a circular patch of radius ρ. Then, whichever of the two figures we draw, the points of the other figure come inside the blackened areas of our diagram: the figures are indistinguishable because unless we can apply tests finer than our standard we can not say which of them the diagram is intended to represent.

We must be careful not to underestimate the differences that may exist between two indistinguishable figures, or we shall be in danger of overestimating the difficulty of proving them indistinguishable. We have only to know that the circle of radius ρ round an arbitrary point P of Γ contains at fewest one point of Δ and that the circle of the same radius round an arbitrary point Q of Δ contains at fewest one point of Γ. If the circle round P contains several points of Δ, the equal circles round these points all contain P, and as far as these points are concerned the condition for the indistinguishability of Δ from part of Γ is established by the presence sufficiently near them of this one point P. We are not called upon to set up a correspondence between points of Γ and points of Δ, and, in particular, we have no need to investigate whether a one-to-one correspondence exists.

As an example, let Γ be the circumference of a circle of radius σ, and let Δ consist of one point only, the centre of the circle. Then if the standard of tolerance is greater, or to use the inevitable word is coarser, than σ, the figures are indistinguishable. Any figure whatever which is inside a circle is indistinguishable from a single point inside or on the circle, to take one extreme, or from the totality of points forming the area of the circle, to take the other extreme, by any standard coarser than the length of a diameter.

If two figures are indistinguishable within one standard of tolerance, they are necessarily indistinguishable within any coarser standard, but as a rule there are also standards fine enough to render the figures distinguishable. A circle of radius δ is indistinguishable from an inscribed square within any standard coarser than $\delta(1 - \cos \frac{1}{4}\pi)$, but distinguishable from the square within any standard finer than this. In fact for any two figures the upper bound of the class of distances separating a point of one figure from the whole of the other figure is a length ν such that the figures are indistinguishable within any standard coarser than ν, distinguishable within any standard finer than ν; whether they are distinguishable within the actual standard ν does not matter.

2. Consider now the relation of a circle of radius δ to the regular polygons inscribed in it. The circle is indistinguishable from the triangles within any standard coarser than $\delta(1 - \cos \frac{1}{3}\pi)$, from the squares within any standard coarser than $\delta(1 - \cos \frac{1}{4}\pi)$, from the pentagons within any standard coarser than $\delta(1 - \cos \frac{1}{5}\pi)$, and so on. Whatever standard ρ is proposed, there are values of n such that $1 - \cos(\pi/n) < \rho/\delta$, and therefore there are regular polygons from which the circle is indistinguishable within the given standard. This is what is meant by saying that the circle is a limit of regular polygons inscribed in it.

Are we then at the end of the story already? Not quite. Suppose first that we replace the circumference of which we have just been speaking by a circumference with one point omitted from it. Whatever the standard of tolerance, any figure which is indistinguishable from the complete circumference is equally indistinguishable from the incomplete circumference. The same is true if we omit not one point only but any finite number of points,

and we can even omit an infinite number of points if we are careful not to take away the whole of any arc, however short. If we wish for a sense in which it is the whole circle, not a speckled band, that is to be called the limit, the definition with which we seemed to be in touch is not precise enough.

Again, think of the secants through a point of a curve. Whatever the angle between them, two lines through a point, if they are not identical, diverge until the distance between a point of one of them and the nearest point of the other exceeds any preassigned limit; in other words, whatever the standard, coarse or fine, it is impossible that two intersecting lines should be indistinguishable. Similarly every circle through two points is distinguishable from the line joining those points. Thus, although indistinguishability within an arbitrary standard appeals to us at first as the essence of a limit, this condition does not allow the tangent at P to be a limit of secants through P, or the line AB to be a limit of the coaxal system of circles through A and B.

Thirdly, when we have attached a meaning to the statement that the tangent is a limit of the family of secants, or that the line is a limit of the coaxal system of circles, we have still to analyse the statement that the tangent is the limit of the secant *PQ as Q tends to P*, or that the line is the limit of the circle *as the radius tends to infinity*. You will readily believe that if I neither illustrate the theory which I have to give you by examples, nor do more than touch on the details of logical proofs, it is not that I regard the theory as dubious or difficult, but that time is wanting.

3. For the first difficulty of which I have spoken, the difficulty illustrated by the omission of points from a circle, I have no remedy more impressive than the drastic but unheroic method of simple evasion. If we have a theory that is satisfactory for complete figures, then for school purposes at least we must be content with that. But it is to be noted that when I say that the figures are to be complete, I mean complete in a sense that is topological. I am prepared to deal with a chord AB, without insisting on bringing in the whole of the unlimited line AB, provided that it is understood that every point between A and B is left to me. I am willing to recognise one branch of a hyperbola, and to cut dead the other branch that is its inseparable companion not only in an analytical treatment but also in any satisfactory discussion by pure geometry. In the language of the theory of sets of points—and a figure is simply a set of points—the precise requirement to be made is that every figure which occurs includes all its own limiting points.

Need I explain? If a point P belongs to a set of points Δ, the distance μ separating P from Δ is of course zero. If P does not belong to Δ, the distance may still be zero, for the set may come indefinitely near to P; for example, the separating distance is zero if P is a point on a line and Δ consists of all the points of the line with the single exception of P. But having noticed this possibility when P does not belong to Δ, we recognise the same possibility when P does belong to Δ. The point P may be isolated from the other points of Δ, in which case the zero separation is so to speak accidental, or there may be other points of Δ indefinitely near to P so that the separation remains zero even if the fact that it occurs as the distance of P from itself is not taken into account. To say that a set comes indefinitely near to P implies that any circle round P, whatever its radius, has points of the set inside it, and using this elementary form of words we may say that *a point P is a limiting point of a figure Δ if every circle round P has inside it some points of Δ other than P itself*; the definition does not assume that P belongs to Δ, for if P does not belong to Δ the qualification "other than P itself" merely does not come into operation.

A figure that includes all its own limiting points I call a complete figure. In this sense, lines and circles and conics are complete; so are polygons, whether the sides are supposed to end at the vertices or to be prolonged indefinitely. If any finite number of complete figures are combined, the

complicated figure which they form is complete. In short, all the figures of elementary geometry have this property. Moreover, if the function $\phi(x, y)$ is subjected to the conditions which we take for granted in elementary analysis, the curve given by $\phi(x, y)=0$ is complete. Our discussion of limits is to be restricted by the understanding that all our figures are complete, but this treatment excludes no figures that could possibly at this stage be required in geometry or admitted by analysis.

The restriction to complete figures serves another purpose. In the example of the circle as a limit of inscribed polygons, it is to be noticed that while there are polygons indistinguishable from the circle within any assigned standard, no sooner has a polygon been chosen than we can find a standard fine enough to distinguish that polygon from the circle: if the polygon has n sides, all that is necessary is that the standard should be finer than $\delta\{1-\cos(\pi/n)\}$. If we want a polygon indistinguishable within the new standard, we must increase the number of sides. This is always possible, and the alternation of finer standard and more numerous sides is interminable, but this implies that we are describing a relation of the circle not to one individual polygon but to a whole family of polygons. As the standard is refined, each polygon becomes distinguishable in due course, but undistinguished polygons always remain. This is the essence of a limit in every case: a limit belongs to a family, not to one member of a family. Since this is a result which our definitions must achieve if they are to be satisfactory, they will fail if it is possible for two distinct figures Γ, Δ to be individually indistinguishable within an arbitrary standard, for then Γ will satisfy in relation to any family whatever which has Δ for a member, the definition by which we are proposing to recognise the limits of the family; whatever the standard, the one figure Δ can be compared with Γ, and no relation of any kind between Γ and the family as a whole is involved.

This possibility is easily examined. If P is a point that belongs to Γ but not to Δ, to say that Γ is indistinguishable from Δ within an arbitrary standard ρ implies that there is a point of Δ within the arbitrary distance ρ of P, and this is to say that P is a limiting point of Δ. *Two figures are indistinguishable within every standard of tolerance whatever if and only if every point of each of them either belongs to the other or is a limiting point of the other.* If either of the figures is incomplete, this condition can be satisfied: whatever the standard, the figure obtained by omitting one point from a circle is indistinguishable from the complete circle, although the figures are actually different. But if both figures are complete, the condition says simply that every point of each figure belongs to the other, which is a complicated way of saying that the figures are identical. *If two complete figures are different, there are standards fine enough to distinguish them from each other.* It follows that if a complete figure Γ is so related to a family of complete figures that for every standard a member of the family, different from Γ and indistinguishable from Γ within the standard, can be found, this can not be because any one member of the family is permanently indistinguishable from Γ; the interminable alternation of a refined standard and a new member of the family, which we noticed in the example of the circle and the polygons, is implied by the condition, and this is a relation of Γ to the family, not to any particular member of the family.

4. With the restriction which we are to assume, our tentative suggestion as to the fundamental characteristic of a geometrical limit is, that if a complete figure Γ is so related to a family of complete figures κ that whatever standard of tolerance is laid down there is some member of κ different from Γ but indistinguishable from Γ within the chosen standard, then Γ is a limit of κ. I will not delay to argue that this condition is sufficient; we have already seen that there are cases in which it is certainly not satisfied, and our next task is to examine these cases with a view to framing a more comprehensive criterion.

Fortunately nothing is easier than to detect the root of the trouble. In the case of the tangent to a curve at a point P, it is true that however small the

angle between the tangent and a secant through P the lines must be distinguishable, whatever the standard of tolerance; at the same time, the smaller the angle between the lines, the further from P we must look to find points of one line whose distances from the other line exceed a given amount, and if we were to lay down in advance not only a standard of tolerance but also a distance from P beyond which we did not care whether our lines were distinguishable or not, then we could secure indistinguishability extending to that distance, although the distance would be in the first place as arbitrary as the standard of tolerance. Similarly of the circles through two points A and B; the larger the radius, the less distinguishable is the circle from the line AB not merely between A and B but for any distance we please beyond these points; by increasing the radius we actually increase the distance between the line and the part of the circle diametrically opposite to A and B, but if we have already decided to take no notice of points further from the midpoint of AB than some assigned distance ν, the part of the circle which is not literally negligible is only a fraction of the whole circumference, and as the radius is increased, this fraction bears a smaller proportion to the whole, and becomes less and less distinguishable from the part of the line which extends to the distance of ν in each direction from the midpoint.

The essence of the solution of the problem is in these examples. We have all enjoyed Mr. Hope-Jones' description of infinity as the place where things happen that don't. What I am saying now is, roughly, that in regard to limits, infinity is such a place that the things that happen there don't matter. This form of words can of course be challenged very seriously, but I am sure you do not want me to waste time in considering whether an attempt to express a technical process in semi-popular language is misleading or helpful. Rather must I ask you to bear with me while I make the language more abstract, for there is a slight difficulty to be overcome.

We are to replace the conception of indistinguishability throughout the whole plane by that of indistinguishability throughout a region of the plane. In the discussion, I shall use the words region and figure to remind us of the actual application to be made of our conception, but in using these words I do not wish to imply that any limitations are understood; as far as our definitions are concerned, regions and figures are just sets of points in the plane.

Our first impulse is to say simply that two figures Γ, Δ are indistinguishable within a standard ρ throughout a region Σ if the portions of these figures within the region, that is, the portions composed of points belonging to the region, are indistinguishable within the same standard. But suppose that Γ, Δ are two parallel lines whose distance apart is less than ρ, and that Σ is the region bounded by a square $ABCD$ of which the side AB lies along the line midway between Γ and Δ, and the sides AD, BC intersect Γ in E and F. Regarded as figures in the whole plane, Γ and Δ are indistinguishable within the standard ρ, but within Σ there is a portion of Γ consisting of the line EF while there is no portion at all of Δ. Thus we have a trilemma: we must admit either that figures which are indistinguishable throughout the whole plane may be distinguishable inside limited regions of the plane, or that in order to know whether two figures are distinguishable inside a region we may require to know the forms of the figures outside the region, or else that a figure that does exist may be indistinguishable from one that does not. Expressed in this form the three courses seem equally anomalous. Nevertheless there can be no doubt that it is the third course which must be taken; on the basis of either of the other admissions, the reduction of the definition of indistinguishability to a symbolic form is enough to show that logical argument must be insufferably complicated. Moreover, the anomaly of the third course disappears on examination. Within the standard ρ which is being assumed, the line EF is indistinguishable from the side AB of the square, and therefore as long as we are content with this standard, we ought not to be sure that there is any

part of the line Γ inside the square. To put the matter materially, in asking whether two figures are distinguishable inside a region, it is not consistent to suppose the boundary of the region to be drawn with a finer pencil than the figures themselves, and any part of either figure that is blurred by the boundary of the region should be automatically excluded from further consideration. Actually whatever the region, the boundary is a set that is easily defined, but we can avoid digressing without altering the substance of our definition:

A figure Γ is said to be indistinguishable from part of a figure Δ throughout a region Σ within a standard of tolerance ρ if every point of Γ has within distance ρ of it either a point that belongs to Δ or a point that does not belong to Σ.

Two figures are indistinguishable from each other throughout a given region within an assigned standard if each of them is indistinguishable from part of the other.

5. Our preparations for the definition of a limit in geometry are at an end:

A complete figure Γ is a limit of a family of complete figures κ if whatever standard of tolerance ρ is assumed and whatever region Σ is taken, then provided only that Σ is bounded there is some member of κ different from Γ but indistinguishable from Γ throughout Σ within the standard ρ.

There is one term in this definition on which no comment has been made, namely, the epithet "bounded". To say that a set of points in a plane is bounded, or does not extend to infinity, means of course that if Q is some point in the plane, the distances of the points of the set from Q are all less than some single distance ν: the upper bound of the class of lengths each of which is the distance from Q to some point of the set is finite. If the distances from Q are all less than ν, the distances from any other point R are all less than $|QR|+\nu$, and therefore the condition for a set of points to be bounded does not really involve reference to one point of the plane rather than to another. We may say alternatively that a set is bounded if there is some circle which surrounds the whole set; if there is one such circle, then again such a circle can be described round any centre that is convenient.

It is a consequence of our definitions that if two figures are indistinguishable throughout a region T within the standard ρ, and if Σ is any region which forms part of T, then the figures are indistinguishable throughout Σ within the same standard. The practical importance of this result is that although in our general definition of a limit we do not restrict the form of Σ, in any particular problem we need consider only bounded regions of appropriate forms. For example, if the condition for a limit is satisfied as regards indistinguishability throughout an arbitrary circle round any one point Q, it must be satisfied as regards any bounded region whatever, since any bounded region is by definition inside some such circle. In dealing with conics it is often convenient to take the standard bounded region in the form of a rectangle.

It will be seen that the theory I am describing avoids completely the difficulties which an honest boy feels on being told that a parabola is a limiting form of ellipse or hyperbola. No likeness can be recognised between a closed curve surrounding a region whose width increases to a definite maximum and then decreases steadily and an open curve whose two arms spread indefinitely farther and farther apart. No confusion is possible between the single open curve of the parabola and the pair of open curves which compose one hyperbola. But if we associate with a parabola a bounded region of the plane, the case is altered. We can find ellipses which attain their greatest width outside the chosen region; within this region the width of such an ellipse, like that of the parabola, increases steadily with the distance from the vertex; the inevitable narrowing, and the second crossing of lines parallel to the focal axis, take place, but take place in a part of the plane excluded from consideration. Similarly we can find hyperbolas which have one branch wholly beyond the pale; the distant branches exist, but their existence, in a region where by hypothesis everything is to be ignored, can be of no consequence.

6. Although I have still to describe certain essential developments of the theory, this is a convenient moment for referring to the character of fundamental theorems and to the nature of detailed demonstrations of them. A typical theorem is that if each figure in a family consists of three collinear points, and if a limit of the family consists of three points, the three points composing the limit are collinear. The proof is by establishing a contradiction: if three points A, B, C are not collinear, it is possible to draw round these points three equal circles so small that no line which crosses two of the circles can cross the third; it follows that no figure which consists of three collinear points can consist of three points one in each of these circles, and from this it follows that the figure composed of the points A, B, C must be distinguishable from any figure composed of collinear points if the standard of tolerance is finer than the radius of the three circles; that is to say, the figure formed of A, B, C is not a limit of any family whose members are collinear triads, and any triad which is a limit of such a family is therefore collinear. The theorem is dull and the proof tedious, but the same is true of theorems and proofs at the corresponding stage of analysis. What is important to observe is that the meaning of the theorem is precise and simple, and that if we are challenged to produce a formal proof we have no difficulty in doing so.

Before passing on, I must refer to the misgiving of a friendly critic that the theory of standards of tolerance is remote from the real problem of limits, which is to render rigorous such arguments as are used in proving from the bifocal property of an ellipse that the tangent bisects an angle between the focal radii: if P, Q are two points of the curve, and if the projection of P on SQ is U and the projection of Q on $S'P$ is V, then in the limit $|UQ|$ and $|PV|$ are equal, and therefore in the limit PUQ and QVP are congruent triangles. I have no time to rehabilitate this proof in detail, but the essence of the matter is that an appeal to the shapes of vanishing figures is necessarily an appeal to the geometry of similar figures which are not evanescent. That $|UQ|$ and $|PV|$ are equal in the limit *means*, in this argument, that their ratio tends to unity, and the proof that the angles UQP, QPV tend to numerical equality is to be conducted on triangles which are not infinitesimal, for example, on triangles puq, qvp in which p and q are fixed points. The ideas of this paper do apply, first to the discussion of these larger triangles, and then to the proof that the angles SQP, QPS' can not tend to equality, P being a fixed point, unless PQ tends to a bisector of SPS'.

7. So far we have been dealing only with the geometrical elements in our figures, not with any numerical measurements that we may be accustomed to make. To embrace with as little complication as possible a limit of the eccentricities of the conics of a family or a limit of cross-ratios, we may suppose that we express all lengths in terms of a unit length λ. A standard of tolerance then becomes a signless number ϵ; instead of saying that a distance between a point of one figure and some point of another is to be less than ρ, we have only to say that it is to be less than $\epsilon\lambda$, and at the same time we can postulate that differences between measurements on one figure and corresponding measurements on another are to be less numerically than ϵ. Similarly, instead of saying that all points at greater distances than P from some fixed origin are to be ignored, we say that they are to be ignored if the distances are greater than $\mathrm{E}\lambda$, where E is again a signless number, and then we can remark of a figure that a particular measurement is greater than E. If by a measured figure we mean one on which a number of measurements have been made—not, of course, one on which every conceivable measurement has been made—we can say that a measured figure Γ is a limit of a class of measured figures κ if given a fixed origin O, and any two numbers ϵ, E, there exists some member Ξ of κ distinct from Γ and such that within a circle of radius $\mathrm{E}\lambda$ round O the figures are geometrically indistinguishable within the standard of tolerance

$\epsilon\lambda$, while each algebraical measurement on Γ which is less than E differs from a corresponding measurement on Ξ by less than ϵ.

It will be noticed that with the measurements of the figures there is a qualification that we have avoided throughout the previous work. In treating geometrical figures we have not used any assumption that a part of one figure is recognised as corresponding to a part of another; since it is not possible to define an infinite number of figures individually, some correspondence is in fact implicit in the admission that we have infinite families to deal with, but we have not had to refer to this correspondence.

8. It is, however, not only in connection with measurements that correspondences must be traceable: the familiar language that one part of a figure tends to one position while another part tends to another involves the same discrimination.

An attempt to analyse the properties of the correspondence that is to be assumed would be superfluous. In all elementary work the composition of the variable figure, that is to say, of the typical figure of the family whose limits are to be discussed, is described in terms in which whatever correspondence we require is implicit if not explicit. We speak, for example, of the polars of two fixed points A, B for a variable circle; this means that we have a family of figures, each consisting of the points A, B, a circle, and two lines, and it goes without saying that in comparing any two figures of the family we are comparing the polar of A in one with the polar of A in the other. This is a comparison between the lines themselves, not a correlation between points on one line and points on the other; further details regarding the figure may imply attention to further correlations, as for example between the points in which the polars of A are cut by some fixed line, or between the points in which the polars of A and B intersect.

Suppose then that in the typical member Ξ of a family of figures κ, we can recognise two constituents, Θ, Φ, which do not necessarily make up the whole figure and which may have points in common. Suppose that the construction of Ξ is such that when Θ is given, Ξ is determined. Suppose that in a figure Γ which is a limit of κ we can recognise constituents Δ, H which correspond to Θ, Φ, and suppose lastly that there is no limit of κ in which the precise figure Δ corresponds to Θ, but H does not correspond to Φ. Then we say that as the part Θ of the variable figure Ξ tends to Δ, the part Φ tends to H.

The assumptions we have enunciated, and the necessity under which the last of them puts us of examining all the limits of κ in which Δ corresponds to Θ, sound formidable in theory, but in practice they seldom lead to difficulties. Not that κ has only one limit. On the contrary, in all the elementary applications of the theory every member of κ is a limit of κ: every secant through a point P is a limit of the family of secants through P, every ellipse with a given focus and directrix is a limit of the family of ellipses with the same focus and directrix. But when we appeal to the theory of limits it is usually because we want to deal with a limit which does not belong to the family with which we wish to associate it: we propose to derive properties of the tangent from those of secants, properties of the parabola from those of ellipses. In elementary geometry it often happens that there is only one limit which does not belong to the family, or at worst that such limits are few in number and are readily distinguished from each other: if P is a point on an ellipse, the tangent at P is the only limit of the family of secants which is not itself a secant; if P is a point on a hyperbola, the family of secants has three special limits, the tangent at P and the lines through P parallel to the asymptotes; the family of ellipses with a given focus and a given directrix has two limiting figures which are not ellipses, namely, the parabola, and the figure of which the focus is the only point. Thus problems of selection and identification are apt to be trivial.

One detail in the definition of the conditions under which Φ tends to H as Θ tends to Δ deserves notice. Since Θ determines Ξ and Ξ is typical of an infinity of figures, Θ also is a typical member of an infinite family, and Δ is a limit of this family. As a rule, Φ also is a typical member of a family of which H is a limit, but there is nothing in the definition actually to prevent Φ from being an invariable constituent of Ξ, in which case this constituent is of course part of Γ and H coincides with it.

9. This consideration tempts me to conclude with a reference to pure analysis, where the technical difficulty of framing definitions in such a way that a dependent variable whose value is constant does not need special discussion is familiar: there are books according to which a series ceases to have a sum if it terminates, even though zero terms are formally introduced. The suggestion we derive incidentally from our geometrical discussion is, that in a case, to take an example, of dependence of a variable w on a variable z, we are to say that w tends to b as z tends to a, if the pair of numbers (z, w) has the pair of numbers (a, b) for a limit, and if there is no pair of numbers (a, c) with c different from b which is a limit of the pair (z, w); the independent variable z can not be always equal to a, and so we can require in any neighbourhood of (a, b) a pair (z, w) distinct from the pair (a, b) even if w is always equal to b. But I should not press this point of view for the sake only of an incidental advantage. It seems to me that a far more fundamental question is involved. The teaching of analysis began to be rigorous before the importance of the theory of sets of points was appreciated, and we have not to go back many years to find text-books where there is a scrupulous treatment of functional dependence, but no mention whatever of aggregates. Nowadays the points of accumulation, or limits, of an aggregate, are usually defined before anything is said of variables and functions, but the definition of functional limits generally follows the classical lines and is entirely independent of this preliminary work on aggregates. Undoubtedly the whole subject gains in unity if the conception of a functional limit is derived first from that of the limits of a multidimensional aggregate. This was in essence the course followed as long ago as 1907 by Veblen and Lennes in their *Introduction to Infinitesimal Analysis*. The teacher who reconstructs such a course for himself may wish to be assured that he is right in inferring the classical *criterion* for a limit of a function from Weierstrass' theorem on the *existence* of a limit of an aggregate: the inference is a simple corollary from this theorem, and is not otherwise to be drawn.

E. H. Neville.

Discussion.*

The **Chairman** (Mr. Fletcher) thought most of those present would be glad of an opportunity of digesting the paper before venturing to say anything about it. Personally, he had expected to find the whole subject out of his ken and quite repulsive to him, but he had found himself on very much more familiar ground than he anticipated and had been more interested than he expected. There was obviously another point which underlay the whole thing. People professed to establish mathematical things on experimental results. Experimental results went, at the very outside, in the most favourable cases, to say seven figures, and yet it was professed to establish on these things that were absolutely true. That was impossible. Even when an astronomer used to say—he did not say it in these days—that a body moved in an ellipse, he was going far and away beyond his data. His data went to two or three figures, and yet the astronomer pretended to predict a thing as absolute truth that would go on for ever and ever. There was a great deal in the subject, given a little understanding, that would appeal to the most commonplace-minded amongst Professor Neville's audience. He hoped none would leave thinking it was high-falutin stuff that did not concern them, because it really did. He

* At the Annual Meeting of the Mathematical Association, January 6th, 1931.

did not say it directly concerned their everyday teaching, but there was much that could be taken advantage of.

Professor **D. K. Picken** said he had been a member of the Association for twenty-six years, but had never previously attended an Annual Meeting, because he had spent most of these years in other parts of the world. It seemed to him that what the Association should be trying to do came out very clearly in the paper to which members had just listened. Surely the important function of the Association was to draw together those teaching in schools with those working in the higher flights of mathematics. All probably recognised the danger of those two sections drifting too far apart, to the detriment of both. The paper just read seemed extraordinarily interesting from that point of view, because it dealt with the concrete commonsense type of limit problem in which all were interested. The problem, for instance, of the relation of a family of ellipses and hyperbolas to the parabola of the system showed the original mathematical worker analysing more closely the commonsense ideas that all of us work with. If there could be that kind of action and reaction going on all the time it would be beneficial to all concerned. For instance, it might be possible to say to Professor Neville that, as the next step in dealing with the question, it would be a very good thing indeed if he would work at the closing of the gap between the old intuitive approach and his own highly original approach to the same problem. There is much to be done in pulling those two ends together, from the point of view of developing commonsense grasp of the facts—one of the most important things that teaching could do.

There was one further point in relation to problems like that of the ellipses and the parabola, which was of special interest. There was a danger in the kind of work under discussion, as in all modern analytical work, that it would take away from us some of our mathematicians' fairy-land. After all the infinite, especially in geometry but also in analysis, was the mathematicians' fairy-land, and it must not be sacrificed to the modern spirit. If it were, then the mathematician would do both himself and the community great harm, because this question of the infinite was of ever increasing importance in philosophic thought, and those who worked on the mathematical approach had the key to it. The pure philosopher fumbled about with it, whereas the mathematical worker knew a great deal about it; but he also knew the mystery that was always bound up with it. It seemed necessary to warn the pure analyst not to whittle those mysteries down; they were essential, and perhaps the most important facts with which the mathematician had to deal.

The **Chairman**, in closing the discussion, said the Association had had two exceedingly good meetings during the past two days, and Professor Neville had given a most effective wind-up. He was sure all present were very much obliged to him. (Applause.)

Professor **Neville** thought any reader of a paper who found he had so large an audience at 5 o'clock in the afternoon had every reason to feel flattered. He could assure Professor Picken that although he confined his standards of tolerance to the finite domain he was not intolerant as regards infinity!

The proceedings then terminated.

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805. Another headache for the relativists. "A motor-car going fast occupies more space, not less, than a motor-car going slow."—*Punch*, July 23rd, 1930. [Per Mr. James Buchanan.]

806. Headline from *The Evening Standard*, July 25th, 1930:

"Bradman, c. Duleepsinhji, b. Peebles 14!"

If Bradman were to make 60 runs an hour, playing 6 hours a day, 6 days a week (allowing 52 such weeks to a year), he would complete his innings in 776,160 years! [Per Mr. James Buchanan.]

MULTIPLICATION AND DIVISION OF INTEGERS AND DECIMALS.

By W. F. Sheppard, Sc.D.

1. *Preliminary.* The arrangement of the working of a multiplication or division sum involving decimals was discussed by the London Branch of the Association in March 1929. Contributions to the discussion, by Mr. R. A. M. Kearney and Mr. F. C. Boon, stating their own methods, will be found in recent numbers of the *Gazette* (see refs. 1 and 2, at end).

My purpose in entering into the discussion is to call attention to three methods which seem worthy of consideration. The first, method (A), is a method of arranging the multiplication or division of an integer by an integer. The second method, (B), suggested in part by (A), but on a different principle, is a method for showing the multiplication or division of a decimal by a decimal. These two methods were propounded by me some years ago (ref. 3), but they seem to be practically unknown.

The methods were not put forward as specially suitable for use in schools. Method (A) is, I think, quite suitable. Method (B) is open, like some other methods, to the objection that it is an *indirect* method: the pupil, when told to perform a certain process, does something else which leads to the same result. Whether this completely bars the method or not, it is, I think, desirable that direct methods should be practised and understood before indirect methods are introduced. I am therefore suggesting a third method, (C), which is, at any rate in form, a direct method. I am not sure that it is easier to work than (B); but it may be found more interesting. The main feature of the method might perhaps be described by Mr. Boon as "tracking the decimal point".

2. *Principles.* The methods are based mainly on the "First thing first" principle: that the pupil's attention should at the beginning of the working be directed to the more important part of the work. This is a matter which is of great importance, if only because it enables the pupil to realise the concrete aspect of what he is doing, instead of merely regarding it as a particular kind of "sum", to be worked out mechanically.

The application of this principle to multiplication, for instance, leads us to begin the multiplication with the digit of highest denomination in the multiplier: the leading part of the product then appears in a prominent position, instead of being relegated to the left-hand bottom corner. Another effect of the application of the principle is that the partial products are placed by means of their initial digits, instead of by their final digits: this has the incidental advantage that it enables us to omit negligible digits in approximate calculation.

3. *Notation.* The notation is the same as in a recent address (ref. 4, § 19). There is a number U, which is our *unit*: when it is taken q times, the result is $P \equiv qU$. I call q the *quotient*, and P the *product*. If the unit and the quotient are given, we find the product by multiplication: if the unit and the product are given, we find the quotient by a "measuring" division: if the quotient and the product are given, we find the unit by a "sharing" division. It is, of course, immaterial whether we multiply U by q, or q by U; but we usually take the simpler of the two to be the quotient. We can then regard U as the numerical part of an arithmetical quantity, so that it is really this quantity that we multiply by q. I do not think confusion can arise from our using the same terms, "unit" and "quotient", whether we are doing multiplication or division: but, as the quantity to be divided is not always qU but may be qU *plus* a remainder, it is sometimes clearer, in division, to use "dividend".

The three numbers U, q, and P (without the remainder), and 1 ("unity"), are the four terms of a proportion relation, which can conveniently be expressed diagrammatically, as in (1),

(1)

$$\begin{array}{c|c} 1 & U \\ \hline q & P \end{array}$$

or in some similar way. The "1" may usually be omitted.

In the numerical examples, the unit and the product (or dividend) are usually shown in ordinary type, the quotient in *italic* type.

The sign × means "times", not "multiplied by".

I abbreviate "partial product" to "p.p."

(A). Multiplication or Division of Integer by Integer.

4. *Multiplication.* I begin with multiplication of an integer by an integer We want to multiply* 42753 by 534. The process, in its first stage, is shown in (2). When this is understood, we can omit the guiding entries on the left-hand side and the terminal 0's on the right-hand side, and proceed as shown

(2)

$$\begin{array}{r|cccccccc} & 4&2&7&5&3&&& \\ \hline \mathit{500} & 2&1&3&7&6&5&0&0 \\ \mathit{30} & &1&2&8&2&5&9&0 \\ \mathit{4} & &&1&7&1&0&1&2 \\ \hline \mathit{534} & 2&2&8&3&0&1&0&2 \\ \hline\hline \end{array}$$

(3)

$$\begin{array}{cccccccc} 4&2&7&5&3&&& \\ \mathit{5}&\mathit{3}&\mathit{4}&&&&& \\ \hline 2&1&3&7&6&5&& \\ &1&2&8&2&5&9& \\ &&1&7&1&0&1&2 \\ \hline 2&2&8&3&0&1&0&2 \end{array}$$

(4)

1 —	42753
2 —	85506
3 —	128259
4 —	171012
5 —	213765
⋮	

in (3). The unit U is written down first, and the quotient q is written underneath it; the first figure of the latter being vertically beneath the first figure of the former. The partial products—the products of the unit by the successive constituents or figures of the quotient—are entered in the middle section of the diagram. They may be supposed to be taken from a table of multiples of U, such as (4). The initial figure of each p.p. is, in this particular example, placed vertically beneath the figure of the quotient to which it corresponds; the successive initial figures therefore lie in a diagonal line sloping down from left to right.

It will be noticed that the arrangement implies that simple multiplication is to be shown as in (5), not as in (6); also that the work is limited by a

(5)

$$\begin{array}{cc} 4 & \\ 5 & \\ \hline 2 & 0 \\ \hline\hline \end{array}$$

(6)

$$\begin{array}{cc} & 4 \\ & 5 \\ \hline 2 & 0 \\ \hline\hline \end{array}$$

vertical line, from which the unit, the quotient, the first p.p., and the total product, all start.

5. *Initial 0's.* The above is plain sailing. But there is a difficulty with regard to the p.pp. I have said that, in the particular case, the initial figure of each p.p. is placed vertically beneath the corresponding figure of q, and that these initial figures lie in a straight line. If this were so in all cases, we

* The calculations, throughout, are exact. The results must, of course, be conditioned by the degree of accuracy in the particular case.

should have a ready means of seeing whether the p.pp. are correctly placed. But the test, as stated, may fail. If, for instance, we multiply 42753 by 231, we get (7); and the initial figures of the p.pp. are not in a straight line.

(7)

$$\begin{array}{|r}
42753 \\
\mathit{231} \\
\hline
85506 \\
128259 \\
42753 \\
\hline
9875943 \\
\hline\hline
\end{array}$$

(8)

$$\begin{array}{ll}
\mathit{1} \text{ ——} & 042753 \\
\mathit{2} \text{ ——} & 085506 \\
\mathit{3} \text{ ——} & 128259 \\
\mathit{4} \text{ ——} & 171012 \\
\mathit{5} \text{ ——} & 213765 \\
\vdots &
\end{array}$$

(9)

$$\begin{array}{|r}
42753 \\
\mathit{231} \\
\hline
085506 \\
128259 \\
042753 \\
\hline
09875943 \\
\hline\hline
\end{array}$$

What are we to do in such a case? Are we to go back on our principles, and resign ourselves to fixing the p.pp. by their tails instead of by their heads?

The reason of the difficulty is that one or more of the p.pp. have only five figures instead of the six that we should expect. The difficulty is removed by the simple expedient of inserting an initial 0 wherever a p.p. is short of a digit. In other words, we adopt the principle that *the product of a number of* m *figures by a number of* n *figures is a number of* m + n *figures.* The table of multiples thus becomes (8); and (7) becomes (9).

6. *Division.* We now come to division. As before, we shall think of the dividend as the numerical part of an arithmetical quantity, of the same kind as U; so that q is a pure number, and the process is a measuring division. We want to divide 9876789 by 42753. How are we to proceed?

From the principle stated in § 5, that the product of a number of m figures by a number of n figures is a number of $m+n$ figures, it follows that *when a number of* p *figures is divided by a number of* m *figures the quotient is a number of* p − m *figures.* But the dividend in the present case has seven figures, and the unit has five figures: and we know that their quotient has three figures. What are we to do?

Obviously we must take the dividend to be 09876789. The process of division then follows its ordinary course, as in (10), or, more briefly, in (11); subject to the condition that each p.p. contains six figures.

(10)

$$\begin{array}{l|r}
 & 42753 \\
\hline
q & 09876789 \\
\mathit{200} & 08550600 \\
\hline
q - \mathit{200} & 1326189 \\
\mathit{30} & 1282590 \\
\hline
q - \mathit{230} & 043599 \\
\mathit{1} & 042753 \\
\hline
q - \mathit{231} & 00846 \\
\hline
\end{array}$$

(11)

$$\begin{array}{|r}
42753 \\
\mathit{231} \\
\hline
09876789 \\
085506 \\
\hline
132618 \\
128259 \\
\hline
043599 \\
042753 \\
\hline
00846 \\
\hline
\end{array}$$

It will be noticed that the initial figure of each remainder is one place further to the right than that of the previous remainder. Any deviation from this rule might be due to a mistake, or to occurrence of a 0 in the quotient (§ 8).

7. *Criterion for initial 0.* How do we know when we have to prefix a 0 to the dividend?

From the principle we are using, it is easy to deduce the rule that *when two integers are multiplied together the initial figures of the product are less than those*

of either of the factors. For example, 2 × 6 = 12, 2 × 4 = 08; and 12... and 08... are both of them less than 2.... Hence we get the rule that *if the initial figures of the dividend are greater than those of the unit we must prefix a 0 to the dividend.* Thus, if we have to divide 9876789 by 42753 we must make the initial figures of the former less than those of the latter by changing the former to 09876789.

8. *Special case: zero in quotient.* The presence of a 0 in the quotient is a frequent cause of error, especially in division. For multiplication, the strictly correct method would be as in (12); but the use of a special symbol 0̸, representing the zero p.p. due to a 0 in the quotient, enables us to shorten the work, as in (13). The uncrossed 0 in 0̸042753 is, of course, an initial 0. For division, in the same way, (14) gives the strict method, which can be shortened as in (15). The third dividend in (15) is 004398. We enter *0* in the quotient, cross out the first 0 of 004398, and add a 7 at the end of it.

(12)

```
| 4 2 7 5 3
| 2 3 0 1
|-------------------
| 0 8 5 5 0 6
|   1 2 8 2 5 9
|     0 0 0 0 0 0
|       0 4 2 7 5 3
|-------------------
| 0 9 8 3 7 4 6 5 3
```

(13)

```
| 4 2 7 5 3
| 2 3 0 1
|-------------------
| 0 8 5 5 0 6
|   1 2 8 2 5 9
|     0̸ 0 4 2 7 5 3
|-------------------
| 0 9 8 3 7 4 6 5 3
```

(14)

```
| 4 2 7 5 3
| 2 3 0 1
|-------------------
| 0 9 8 3 7 5 8 8 7
| 0 8 5 5 0 6
|-------------
|   1 2 8 6 9 8
|   1 2 8 2 5 9
|   -----------
|     0 0 4 3 9 8
|     0 0 0 0 0 0
|     -----------
|       0 4 3 9 8 7
|       0 4 2 7 5 3
|       -----------
|         0 1 2 3 4
```

(15)

```
| 4 2 7 5 3
| 2 3 0 1
|-------------------
| 0 9 8 3 7 5 8 8 7
| 0 8 5 5 0 6
|-------------
|   1 2 8 6 9 8
|   1 2 8 2 5 9
|   -----------
|     0̸ 0 4 3 9 8 7
|       0 4 2 7 5 3
|       -----------
|         0 1 2 3 4
```

It should be noticed, both in (13) and in (15), that when a 0̸ occurs the diagonal line of initial figures of p.pp. comes to an end, and a new one begins, one place to the right. But in (12) and (14) the line continues.

Multiplication or Division of Decimal by Decimal.

First Method (Method (B)).

9. *Multiplication.* I have considered the case of integers pretty fully. I now pass, omitting intermediate stages, to the problem of multiplying a decimal by a decimal. We want to multiply 427·53 by 23·1. How are we to proceed?

The principle of the method now to be considered is essentially the same as that of the ordinary "standard form" method: that either factor may be multiplied or divided by a power of 10, provided that this is corrected by a corresponding division or multiplication later on. In the ordinary method one of the factors is altered to a standard form, and the corrective process is

performed on the other factor: in my method each factor is altered to a standard form, and the corrective process is performed on the product.

10. The detail of the method is suggested by comparison of the statements $2 \times 6 = 12$, $2 \times 4 = 08$, with the statements $\cdot 2 \times \cdot 6 = \cdot 12$, $\cdot 2 \times \cdot 4 = \cdot 08$. The form which I adopt as the standard one is that of a number between 0·1 and 1·0, *i.e.* a number whose first significant figure follows immediately after the decimal point. There are really nine different cases, according as each of the two factors (*a*) is $>1{\cdot}0$, or (*b*) is between 0·1 and 1·0 (*i.e.* is already in our standard form), or (*c*) is $< 0{\cdot}1$; but we need not consider each case separately. I will take one case, and then consider the general rule.

(i) First suppose that each factor is $>1{\cdot}0$. We want to multiply 427·53 by 23·1. We divide the first factor by 10^3, and the second by 10^2, getting ·42753 and ·231. We multiply these together, as in (16)—which, except for the presence of the decimal points, is exactly the same as (9)—and we obtain the product ·09875943. We have divided one factor by 10^3, and the other by 10^2; we have therefore to multiply the product by 10^5, which gives us 09875·943. We can then drop the initial 0.

(16)

```
·4 2 7 5 3
·2 3 1
----------------
 0 8 5 5 0 6
   1 2 8 2 5 9
     0 4 2 7 5 3
----------------
·0 9 8 7 5 9 4 3
```

(ii) The above leads easily to the general rule, which may be stated as follows.

(*a*) In any number, whether integer or decimal, let k be the number of significant figures before the decimal point; or, if there are no such figures, let $-k$ (k being here 0 or negative) be the number of consecutive 0's immediately following the decimal point. Then k (whether positive or 0 or negative) will be called the *order* of the number.

(*b*) Let the order of the unit be x, and let that of the quotient be y.

(*c*) Let U' be the number, of order 0, which has the same digits as U; and let q' be the number, of order 0, which has the same digits as q. Multiply U' by q', and let the product be P'. Then the required product qU is the number, of order $x+y$, which has the same digits as P'.

(iii) In applying the rule, an initial 0 in P' must be counted as a significant figure. Thus, in the example in (i), $x=3$, $y=2$, so that $x+y=5$; and P' is ·09875943, the 0 in which is, for this purpose, a significant figure. To obtain five significant figures, the decimal point must be placed after the 09875.

(iv) For another example, take $\cdot 00000534 \times 42{\cdot}753$. The product of ·42753 by ·534 is (cf. example (3)) ·22830102; and the order of P is $-5+2=-3$. The product is therefore ·00022830102.

11. *Division.* Division is performed on the same principle: regard being had to the two points (1) that the initial figures of the dividend are less than those of the unit, and (2) that 0's after the decimal point in the unit correspond to significant figures before the decimal point in the quotient, and conversely. Thus to divide ·009876789 by ·00042753 we must write the former as ·0 09876789, and then divide ·09876789 by ·42753. The working is shown in (17). The quotient is of order $(-1)-(-3)=2$; it is therefore 23·1.

(17)

```
·4 2 7 5 3
·2 3 1
----------------
·0 9 8 7 6 7 8 9
 0 8 5 5 0 6
----------------
   1 3 2 6 1 8
   1 2 8 2 5 9
----------------
     0 4 3 5 9 9
     0 4 2 7 5 3
----------------
       0 0 8 4 6
```

Second Method (Method (C)).

12. As I stated at the beginning of this paper, method (B) is open to objection on the ground that it is an indirect method—that we get at our result by doing something different from what we are told to do. But that is not a permanent objection: it is only an objection to the too early appearance of the method. It is quite possible for the pupil to begin with a direct method—Mr. Boon's, for example—and take up a more formal method later, when he is ripe for it.

The general principle of my new method (C) is that the p.pp. are, in the early stages of the subject, to be calculated directly and independently and placed in the scheme in their proper relation to the digits of the quotient; *i.e.* with the initial figure of each p.p. vertically beneath the corresponding figure of the quotient. It will then be found that their decimal points are in a vertical line, and that the positions of the decimal points of the unit, the quotient, and the product (or dividend) are related by the rule given in § 10. This rule may be briefly stated as follows:

The number of significant figures before the decimal point of a product is the algebraical sum of the numbers of significant figures before the decimal points in the factors: a 0 after the decimal point but before the significant figures being counted as (− 1) *significant figures;*

or, more briefly still:

The order of a product is equal to the sum of the orders of its factors.

This rule having been established, we can apply it to finding the position of the decimal point in the product (in multiplication) or in the quotient (in division).

For illustration of the rule, I have taken some examples from Kearney and Boon, and arranged the figures in accordance with my method. The process is quite simple; but it will be seen that modifications of the normal method are helpful when a term contains several 0's after the decimal point.

13. *Multiplication.* In multiplication there are four cases that may be considered.

(i) Ordinary case; both factors being of positive order (§ 10). To multiply 329·71 by 43·08 (Kearney, p. 106). This is done in (18). The factors are written down as they stand, including the decimal points. They are respectively of orders 3 and 2, so that the first p.p. will have five figures before

(18)

```
| 3 2 9·7 1
| 4 3·0 8
|------------------
| 1 3 1 8 8·4
|   0 9 8 9·1 3
|     Q 2 6·3 7 6 8
|------------------
| 1 4 2 0 3·9 0 6 8
```

the decimal point, and the decimal points of the p.pp. and of the total product will be in a vertical line. As to the Q, see § 8. As the pupil gains confidence, the decimal point can be omitted from the p.pp., its entry in the total product being sufficient. This plan is adopted in the following examples.

(ii) *Multiplication of large number by small.* To multiply 786·2 by ·000127 (Kearney, p. 106). The complete calculation is shown in (19), the Q's in the

(19)

$$\begin{array}{|l}
7\ 8\ 6{\cdot}2 \\
{\cdot}\mathit{0\ 0\ 0\ 1\ 2\ 7} \\ \hline
0\ 0\ 0\ 0\ 7\ 8\ 6\ 2 \\
1\ 5\ 7\ 2\ 4 \\
5\ 5\ 0\ 3\ 4 \\ \hline
0\ 0\ 0{\cdot}0\ 9\ 9\ 8\ 4\ 7\ 4 \\ \hline\hline
\end{array}$$

(20)

$$\begin{array}{r|l}
 & 7\ 8\ 6{\cdot}2 \\
\boxed{{\cdot}\mathit{0\ 0\ 0}} & \mathit{1\ 2\ 7} \\ \hline
 & 0\ 7\ 8\ 6\ 2 \\
 & 1\ 5\ 7\ 2\ 4 \\
 & 5\ 5\ 0\ 3\ 4 \\ \hline
 & {\cdot}0\ 9\ 9\ 8\ 4\ 7\ 4 \\ \hline\hline
\end{array}$$

middle section representing the 0's in the quotient. The product is of order $3+(-3)=0$, so that the decimal point is placed before the 0998... of the product.

We can make the work more compact by sliding the ·000 of the quotient out of the framework, as shown in (20). These three 0's balance the three significant figures 786 of the unit, and the net total number of significant figures in the product is 0.

(iii) *Multiplication of small number by large.* To multiply ·00683 by 572·1 (Kearney, p. 106). This is done in (21). But it is better to slide out the two 0's of the unit, as in (22). The order of the product is $-2+3=1$.

(21)

$$\begin{array}{|l}
{\cdot}0\ 0\ 6\ 8\ 3 \\
\mathit{5\ 7\ 2{\cdot}1} \\ \hline
0\ 0\ 3\ 4\ 1\ 5 \\
0\ 0\ 4\ 7\ 8\ 1 \\
0\ 0\ 1\ 3\ 6\ 6 \\
0\ 0\ 0\ 6\ 8\ 3 \\ \hline
0\ 0\ 3{\cdot}9\ 0\ 7\ 4\ 4\ 3 \\ \hline\hline
\end{array}$$

(22)

$$\begin{array}{r|l}
\boxed{{\cdot}0\ 0} & 6\ 8\ 3 \\
 & \mathit{5\ 7\ 2{\cdot}1} \\ \hline
 & 3\ 4\ 1\ 5 \\
 & 4\ 7\ 8\ 1 \\
 & 1\ 3\ 6\ 6 \\
 & 0\ 6\ 8\ 3 \\ \hline
 & 3{\cdot}9\ 0\ 7\ 4\ 4\ 3 \\ \hline\hline
\end{array}$$

(iv) *Multiplication of small number by small.* The method is clear from the above.

14. *Division.* If the unit and the product are respectively of orders x and z, the quotient is of order $z-x$. As with multiplication, we can sort the cases into four categories.

(i) *Ordinary case.* To divide 9379·62 by 371·03 (Boon, p. 158). This is done in (23). Notice that the dividend is 09379·62.

(23)

$$\begin{array}{|l}
3\ 7\ 1{\cdot}0\ 3 \\
\mathit{2\ 5{\cdot}2} \\ \hline
0\ 9\ 3\ 7\ 9{\cdot}6\ 2 \\
0\ 7\ 4\ 2\ 0\ 6 \\ \hline
1\ 9\ 5\ 9\ 0\ 2 \\
1\ 8\ 5\ 5\ 1\ 5 \\ \hline
1\ 0\ 3\ 8\ 7
\end{array}$$

The unit and the dividend are written down first, with their decimal points: room being left for the quotient. The quotient is of order $5-3=2$, so that we can place its decimal point before the division has begun.

(ii) *Division of small number by large.* To divide ·0998474 by 786·2 (Kearney, p. 108). In a case of this kind, where there will be one or more 0's in the quotient, there are two ways of proceeding.

Notice that, since 99... is greater than 78... , the 0 in the dividend is an initial 0, and is therefore to be treated as a significant figure.

(*a*) If we write down the unit, with the dividend beneath it, thus:

$$\begin{array}{l}
\phantom{{\cdot}0\ }7\ 8\ 6{\cdot}2 \\
{\cdot}0\ 9\ 9\ 8\ 4\ 7\ 4
\end{array}$$

we see that the decimal point of the latter is three places to the left of that of the former, so that the quotient is of order $0-3=-3$. We cannot therefore proceed as in (i).

We can, however, introduce three 0's before the decimal point of the dividend, so as to move the dividend to the right, as in (24). The division then proceeds

```
           (24)                                  (25)
     | 7 8 6·2                          ·0 0 0 | 7 8 6·2
     | ·0 0 0 1 2 7                            | 1 2 7
     | 0 0 0·0 9 9 8 4 7 4                     | ·0 9 9 8 4 7 4
     | 0 0 0 0 7 8 6 2                         | 0 7 8 6 2
     |       2 1 2 2 7                         |   2 1 2 2 7
     |       1 5 7 2 4                         |   1 5 7 2 4
     |         5 5 0 3 4                       |     5 5 0 3 4
     |         5 5 0 3 4                       |     5 5 0 3 4
```

in the ordinary way; the quotient begins with three 0's after the decimal point, the first significant figure being 1.

(*b*) Alternative method. Since the order of the quotient is -3, we can place ·000 behind the boundary line, as the beginning of the quotient; the subsequent work, in (25), is the same as in (24).

(iii) *Division of large number by small.* To divide 3·907443 by ·00683 (Kearney, p. 108). We move the ·00 of the unit to the left; the order of the quotient is then seen to be $1-(-2)=3$. See (26).

```
           (26)                                  (27)
 ·0 0 | 6 8 3                             ·0 0 | 6 8 3
      | 5 7 2·1                             ·0 | 5 7 2 1
      | 3·9 0 7 4 4 3                   ·0 0 0 | 3 9 0 7 4 4 3
      | 3 4 1 5                                | 3 4 1 5
      |   4 9 2 4                              |   4 9 2 4
      |   4 7 8 1                              |   4 7 8 1
      |     1 4 3 4                            |     1 4 3 4
      |     1 3 6 6                            |     1 3 6 6
      |       0 6 8 3                          |       0 6 8 3
      |       0 6 8 3                          |       0 6 8 3
```

(iv) *Division of small number by small.* To divide ·0003907443 by ·00683 (variant of the above). The quotient is of order $-3-(-2)=-1$. We can arrange the work as in (27). We take out ·00 from the unit and ·000 from the dividend; and place ·0 in the quotient's line, to make the numbers balance. The work then proceeds as in division of integers.

By this time, however, the directness of the work is nominal. We have, in fact, got round to method (B).

References.

1. F. C. Boon, "Decimal Processes: 'Tracking the Unit'", *Mathematical Gazette*, xv (July 1930), 156-159.

2. R. A. M. Kearney, "Multiplication and Division of Decimals", *Mathematical Gazette*, xv (May 1930), 103-110.

3. W. F. Sheppard, "Arithmetic", *Encyclopaedia Britannica*, 11th ed. (1910), ii, 523-542 (§§ 101, 106-7, 111-2). The article has not been reproduced in the new (14th) edition.

4. W. F. Sheppard, "Variety of Method in the Teaching of Arithmetic", *Mathematical Gazette*, xiv (July 1929), 477-489.

W. F. Sheppard.

MATHEMATICAL NOTES.

989. [K¹. 6. a.] *To Inscribe a Square in Two Circles.*

It is required to describe a square so that two of its vertices lie on one circle and two on another.

It is plain that a pair of adjacent vertices must lie on each circle, and as the join of the mid-points of two opposite sides is perpendicular to them, the square is symmetrical with respect to the line of centres of the circles.

Take the equations of the two circles to be

$$x^2+y^2-2gx+c=0, \quad x^2+y^2-2Gx+c=0.$$

Let the coordinates of a pair of opposite diagonal points be (x_1, y_1), $(x_2, -y_1)$, the latter lying on the second circle.

Then as the sides of the square are equal, $x_1-x_2=2y_1$.

Hence $(x_1-2y_1, -y_1)$ lies on the second circle.

On substitution, we find that the vertex (x_1, y_1) is a point of intersection of the first circle and the curve

$$x^2-4xy+5y^2-2Gx+4Gy+c=0.$$

This is an ellipse whose centre is at the centre of the second circle. On referring it to parallel axes through the centre, we find its equation is

$$x^2-4xy+5y^2=G^2-c=R^2,$$

where R is the radius of the second circle. When, further, the axes are rotated through an angle $\pi/8$ the equation becomes

$$x^2(3-2\sqrt{2})+y^2(3+2\sqrt{2})=R^2.$$

The principal semi-axes of this ellipse are $R(\sqrt{2}+1)$, $R(\sqrt{2}-1)$.

Thus we obtain the following result:

Suppose an ellipse drawn concentric with a circle of radius R and with major and minor semi-axes $R(\sqrt{2}+1)$, $R(\sqrt{2}-1)$. Then if any circle is drawn with its centre on a diameter of the first circle inclined at an angle $\pi/8$ to the major axis of the ellipse, it will be cut by the ellipse in points which are vertices of squares inscribed between the two circles.

H. V. Mallison.

990. [K¹. 3. c.; X. 6.] *A Proof of Euc. I. 47, with application of the method to an explanation of Amsler's Planimeter.*

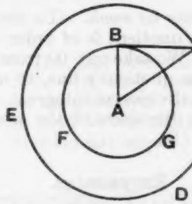


Fig. 1.

ABC is a triangle right-angled at *B* and circles are described centre *A* and radii *AB* and *AC*.

Revolve the triangle *ABC* about *A* through a complete revolution back again to its original position.

As *B* moves round the circle *BGF* it will impart to *BC* a two-fold motion, viz.:

(1) One of revolution about *B* so that *BC* (like the moon going round the earth) revolves once about *B* while *B* itself completes a whole revolution round the circle *BGF*, and hence *BC* traces out an area $\pi \cdot BC^2$ due to this motion.

(2) A rectilinear motion which, since BC is a tangent to the circle BFG, is at any and every instant in a direction BC along its own length and hence traces out no area due to this motion.

Therefore area between two circles = area traced out by BC

$$= \pi BC^2 + 0 = \pi \cdot BC^2.$$

Now area of circle CDE = area of circle BGF + area between two circles.

$$\text{Hence } \pi \cdot AC^2 = \pi \cdot AB^2 + \pi \cdot BC^2;$$

$$\therefore AC^2 = AB^2 + BC^2.$$

Now let CAB represent an Amsler planimeter and let B revolve round the shaded area.

Then shaded area = (area traced out by AB as B moves from B_1 to B_2 along the line B_1BB_2) − (area traced out by AB as B moves from B_2 to B_1 along the boundary line B_2DB_1) = resultant area traced out by AB as B goes completely round the shaded area.

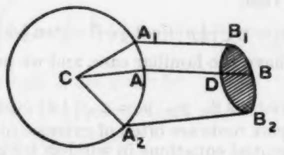


FIG. 2.

Now at any and every instant the motion of AB may be resolved into (1) a motion of rotation about A, and a rectilinear motion which may be resolved into (2) one along AB and (3) one perpendicular to AB.

The resultant area traced out by AB will be the same whether these three component motions be simultaneous or successive.

Taking the latter alternative, the resultant area traced out by AB due to (1) rotation about $A = 0$, since AB returns to its original position; (2) its motion along $AB = 0$, since it is along its own length; (3) its motion perpendicular to AB is clearly the product of AB into the resultant distance traversed perpendicular to AB, and this latter is measured by the wheel so that if $AB = b$ and S be the distance given by the wheel, the area $= b \cdot S$.

E. J. SCHWARTZ.

991. [A. 3. k.] *The Numerical Solution of Cubic Equations by Circular Functions.*

The cubic, transformed as usual to the form $y^3 + py + q = 0$, is to be compared with one of the three fundamental formulae:

$$\text{(1)} \quad \sinh^3 u + \tfrac{3}{4} \sinh u = \tfrac{1}{4} \sinh 3u,$$

$$\text{(2)} \quad \cosh^3 u - \tfrac{3}{4} \cosh u = \tfrac{1}{4} \cosh 3u,$$

$$\text{(3)} \quad \cos^3 \phi - \tfrac{3}{4} \cos \phi = \tfrac{1}{4} \cos 3\phi.$$

We therefore put

$$y = -z\sqrt{|\tfrac{4}{3}p|} \quad \text{or} \quad y = +z\sqrt{|\tfrac{4}{3}p|}$$

according as q is positive or negative, and we have two classes of the transformed equation depending upon the sign of p:

$$\text{(I)} \quad z^3 + \tfrac{3}{4}z = \tfrac{1}{4}k,$$

$$\text{(II)} \quad z^3 - \tfrac{3}{4}z = \tfrac{1}{4}k,$$

where $\quad k = 3|q|/\sqrt{|\tfrac{4}{3}p^3|},$

and k is positive in every case.

I. Determining u by $\sinh 3u = k$, we have for the real root y_1 of the original cubic $y_1 = \mp\sqrt{|\tfrac{4}{3}p|}\sinh u$. This is the usual form of solution by hyperbolic functions. But it is much more convenient to express y_1 in circular functions, which is easily done by making

$$k = \cot\psi = \tfrac{1}{2}(\cot\tfrac{1}{2}\psi - \tan\tfrac{1}{2}\psi) = \sinh 3u$$

if $\cot\tfrac{1}{2}\psi = e^{3u}$. Then

$$\sinh u = \tfrac{1}{2}\{(\cot\tfrac{1}{2}\psi)^{\frac{1}{3}} - (\tan\tfrac{1}{2}\psi)^{\frac{1}{3}}\},$$

and so $$y_1 = \mp\sqrt{|\tfrac{1}{3}p|}\cdot\{(\cot\tfrac{1}{2}\psi)^{\frac{1}{3}} - (\tan\tfrac{1}{2}\psi)^{\frac{1}{3}}\},$$

according to the sign of q. It will be noted that $\tfrac{1}{2}\psi$ is the complement of the gudermannian of $3u$.

II. (i) When $k > 1$, put

$$k = \operatorname{cosec}\psi = \tfrac{1}{2}(\cot\tfrac{1}{2}\psi + \tan\tfrac{1}{2}\psi) = \cosh 3u,$$

if again $\cot\tfrac{1}{2}\psi = e^{3u}$. Then

$$y_1 = \mp\sqrt{|\tfrac{1}{3}p|}\{(\cot\tfrac{1}{2}\psi)^{\frac{1}{3}} + (\tan\tfrac{1}{2}\psi)^{\frac{1}{3}}\}.$$

(ii) When $k < 1$, we have the familiar case, and we put $k = \cos 3\phi$, implying for the three roots

$$y_1 = \mp\sqrt{|\tfrac{4}{3}p|}\cos\phi,\quad y_2,\ y_3 = \mp\sqrt{|\tfrac{4}{3}p|}\cos(\phi \pm 120^\circ).$$

Equations with complex roots are often of extreme importance, for example in the solution of differential equations in wireless telegraphy. The real root y_1 having been obtained, in cases I and II (i), the complex roots are determinable as

$$-\tfrac{1}{2}y_1 \pm \sqrt{(-p - \tfrac{3}{4}y_1^2)}.$$

Example. $$y^3 - 3y + 14 = 0.$$

Here $$k = 7 = \operatorname{cosec} 8^\circ\, 12'\, 47{\cdot}56'',$$

whence $$\log\tan\tfrac{1}{2}\psi = \bar{2}{\cdot}8561049,$$

$$\log(\tan\tfrac{1}{2}\psi)^{\frac{1}{3}} = \bar{1}{\cdot}6187016,\quad \log(\cot\tfrac{1}{2}\psi)^{\frac{1}{3}} = 0{\cdot}3812984,$$

$$y_1 = -2{\cdot}821640\ldots.$$

This is the result by Bruhns' tables, and it is correct to six figures. By Horner's method I obtain $y_1 = -2{\cdot}82164303\ldots$. A. S. PERCIVAL.

992. [K[1]. 6. a.] *Two formulae in Areal Coordinates.*

Although areal coordinates are not generally suitable for metrical enquiry, formulae involving metrical properties are sometimes required. The two following proofs of essential formulae are rather shorter than those usually given in the text-books and have the technical advantage of avoiding any transference to Cartesian coordinates.

We follow the usual practice of writing the fundamental relation between the tangential coordinates p, q, r of a line, $\Sigma a^2(p-q)(p-r) = 4S^2$, in the form $\{ap, bq, cr\}^2 = 4S^2$, ABC being the triangle of reference.

(i) *The perpendicular distance of the point $x_1 : y_1 : z_1$ from the line*

$$lx + my + nz = 0.$$

The equation of the parallel line through $x_1 : y_1 : z_1$ is

$$(x_1 + y_1 + z_1)(lx + my + nz) - (x + y + z)(lx_1 + my_1 + nz_1) = 0,$$

or the perpendicular distances of the two lines from A are

$$2Sl/\{al, bm, cn\},\quad 2S\lambda/\{a\lambda, b\mu, c\nu\},$$

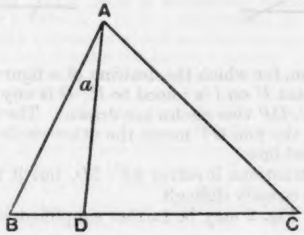
where $$\lambda = l(x_1 + y_1 + z_1) - (lx_1 + my_1 + nz_1),\ \text{etc.}$$

But $$\{a\lambda, b\mu, c\nu\}^2 = \Sigma a^2(\lambda - \mu)(\lambda - \nu) = \Sigma a^2(l-m)(l-n)(x_1 + y_1 + z_1)^2$$
$$= \{al, bm, cn\}^2(x_1 + y_1 + z_1)^2.$$

Hence the perpendicular distance of the point from the line, which is the difference of the distances of the lines from A, is

$$2S[l(x_1+y_1+z_1)-l(x_1+y_1+z_1)+(lx_1+my_1+nz_1)]/\{al,\ bm,\ cn\}(x_1+y_1+z_1)$$
$$=2S(lx_1+my_1+nz_1)/\{al,\ bm,\ cn\}(x_1+y_1+z_1).$$

(ii) *The condition for the lines $p:q:r$, $p':q':r'$, to be perpendicular.*



The parallel lines through A are given by

$$(q-p)y+(r-p)z=0, \quad (q'-p')y+(r'-p')z=0.$$

In the figure, if AD be $my+nz=0$,

$$\frac{c\sin\alpha}{b\sin(A-\alpha)}=\frac{BD}{DC}=-\frac{m}{n}, \quad \therefore \cot\alpha=\frac{mb\cos A-nc}{mb\sin A}.$$

Applying this result to the parallel lines through A, the lines are perpendicular if

$$\{(q-p)b\cos A-(r-p)c\}\{(q'-p')b\cos A-(r'-p')c\}$$
$$+(q-p)(q'-p')b^2\sin^2 A=0,$$

i.e. $(q-p)(q'-p')b^2+(r-p)(r'-p')c^2-bc\cos A\{(q-p)(r'-p')$
$$+(r-p)(q'-p')\}=0.$$

But $(q-p)(r'-p')+(r-p)(q'-p')$
$$=2pp'+qr'+q'r-pr'-p'r-qp'-q'p$$
$$=(r-p)(r'-p')+(q-p)(q'-p')-(q-r)(q'-r'),$$

and the condition becomes

$$\Sigma(q-p)(q'-p')ba\cos C=0, \quad \text{or} \quad \Sigma(q-p)(q'-p')\cot C=0.$$

H. G. Green.

993. [K^1. 21. a. δ.] *On the difficulty of a geometrical construction.*

In a recent article in the *Gazette* (vol. xiv, p. 542) a way of estimating the complexity of a geometric construction was suggested which consisted in counting the number of settings of the ruler or compass (S) and the number of lines or arcs drawn (D).

Examples given were:

to draw a circle centre C to pass through A needs $2S+D$; and
to join AB needs $2S+D$.

It is doubtful if there are any pupils who could consider these two constructions as being equally difficult—if the time taken to perform them be the ultimate test of difficulty. Many other factors, such as proximity and obliquity of lines, should be taken into account.

As a more critical test of the working of the rule we may consider the construction of a line through a given point P parallel to a given line (l).

Figs. 1 and 2 show the usual constructions.

FIG. 1.

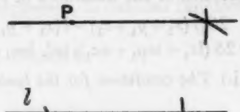


FIG. 2.

Another construction, for which the drawing of a figure is left to the reader, is as follows. Any point U on l is joined to P; O is any point on PU. With centre O and radii OU, OP two circles are drawn. The first of these cuts the line l again in V, and the join OV meets the other circle in two points one of which is on the required line.

Each of these constructions involves $8S+5D$, but it is doubtful if a pupil would regard them as equally difficult.

The construction in Fig. 2 may be further simplified as in 2B and 2C.

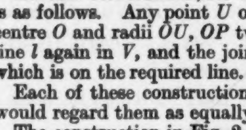


FIG. 2B.

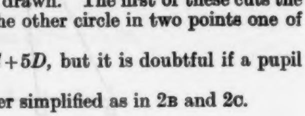


FIG. 2C.

2B. With centre P and any radius draw arc AB. With centre A and same radius draw an arc cutting l in C. With centre C and same radius draw arc at B. Join PB. This needs $5S+4D$.

2C. With centre H any point on l draw arc PK. With centres P and K and the same radius draw arcs cutting in L. Join PL. This needs $6S+4D$. Method 2C, although needing one more setting than 2B, seems to be the more simple.

B. NAYLOR.

Box 438, Bulawayo, S. Rhodesia.

Note on the above.

Miss Naylor does not commit herself as to which of the two operations—to draw an arc centre C through A and to draw the line CA—the pupil would think the easier, but I infer that she means the latter. This may be because she compares the drawing of the whole circle with the drawing of part of the line. However, in most constructions it is only a small part of the circle which has to be drawn. If her construction 2C is found simpler than 2B—I do not personally see that it is so—it would be because a larger portion of the circle has to be drawn in 2B. It is a case of D versus d.

It should be noted that in Lemoine's *Géométrographie* he uses separate letters to represent setting the compass, setting the ruler and so on. But as "other factors such as proximity" should be taken into account, I think my more compact notation will do as well as Lemoine's more elaborate one as a method of estimating, at least roughly, the complexity of a construction.

C. O. TUCKEY.

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807. "My acquaintance with lascars had been so small as to amount to a negative quantity."—*Dark Dealing*, by A. C. Brown (1930), p. 63. [Per Mr. F. P. White.]

REVIEWS.

The Mysterious Universe. By Sir James Jeans. Pp. x+154. 3s. 6d. 1930. (Cambridge University Press.)

What follows is in no proper sense a review of this remarkable and widely read volume. That would be possible only to one who possessed precisely that familiarity with mathematical and physical science which the present writer lacks. But Sir James Jeans' book is something much more than a learned treatise or a popular textbook. It is a description of the physical universe which raises of necessity questions which are the concern of the philosopher and the theologian. In his last chapter Sir James ventures " Into the deep waters ", and there he must be followed, and his argument examined, even by those who, with regard to the body of the book, cannot claim the smallest power to check his facts and conclusions in the strictly scientific sphere. In particular, one whose studies have for many years belonged to the region of theology or to that border-country where theology and philosophy meet may well feel the need of considering whether and how far the picture of the universe given by Sir James Jeans and others affects his own beliefs. And if he is asked not to keep his thoughts to himself he may be right in complying with the invitation.

Three preliminary observations may be made. First, it is clear that on the basis of modern mathematical physics a new phase begins in the relations of science and theology. It will not necessarily prove to be one more favourable to theology (I am inclined to think that some decidedly premature assumptions are being made to that effect), but it means the end of the period when it could plausibly be maintained that theology had received its death-blow from a bluff common-sense materialism.

Secondly, and in close association with the first point, there is the extraordinary appropriateness of Sir James' adjective. A universe which discloses its nature as the result of the application of systems of measurements which deal with " imaginary " numbers like the square root of minus one is mysterious enough almost to deserve the theological word " numinous " as a true description of it. One gets the same impression if one looks not at method but at result. A universe which finally boils down, after the disappearance of one entity and force after another, to " a soap-bubble with irregularities and corrugations on its surface ", as its best representation, is as mysterious as any which has been conceived of by the most speculative of philosophers. Our friend the man in the street will have to get right away from the idea that science, as distinguished from metaphysics and theology, gives him plain concrete facts.

Thirdly, whole tracts of the universe are left outside Sir James' survey. This is an obvious point and is in no way a criticism of the book, except in so far as the sentence on p. 149 as to what has been discovered with regard to the controlling power of the universe is misleading. There it is said that the affinity between our minds and that power does not include " so far as we have discovered, emotion, morality or aesthetic appreciation ". Along the lines of physical and mathematical investigation it would be impossible to discover whether or no anything in common exists as to feeling and morals. The philosopher who hopes that the nature of the universe will throw light upon the nature of God must look at everything in the universe. And there are regions where the relevance of mathematics ceases. Whatever view be taken of art, morals, and religion, at least their reality cannot be dissipated after the manner of the fate which has befallen gravitational forces, and, according to Sir James, is likely to befall electromagnetic forces. In the dissolution of much that was supposed to belong to the material substance of the universe, moral and spiritual values remain as they were.

It would be needless and almost impertinent to enlarge upon the attractiveness of the purely scientific chapters. Their lucidity and their dramatic quality are specially notable. And assuming as one is bound to do that in these pages Sir James Jeans speaks for all those who have gone most deeply into the questions arising in connexion with astronomical physics, one may pass on to his last and definitely interpretative and speculative chapter.

The problem that faces us there is that of creation. Any view of creation must involve conclusions as to the nature of time and eternity. To take this last point first: I suppose that the physicist when he talks about time can leave on one side the relation of time to eternity, in fact that he need not talk about eternity at all. But the philosopher cannot do this; he must try to make something of the relation. I do not find it easy to understand Sir James here. In what sense is it possible that time " from its beginning to the end of eternity " is spread before us in the picture of the universe? In that case, Sir James quotes Weyl as saying, " events do not happen; we merely come across them ". But then, what intelligible distinction can be preserved between time and eternity? And while I do not know how much the word " event ", which has become a very difficult one in the metaphysics of modern science, carries in this passage, the natural interpretation would be that any real freedom is ruled out if Weyl is right. And I think that there is more than a pedantic cavil in the suggestion that such a phrase as " the end of eternity " points to some real confusion here. But I wish to make it clear that Sir James does not commit himself to this hypothesis; in the one which follows, some reality of freedom is preserved.

The universe, as Sir James Jeans conceives it, is, finally, describable in terms of pure mathematics: in the most accurate account which can be given of it the physicist must leave the field to the mathematician. It is a universe of pure thought, in which objective realities exist and may have as their most exact label " mathematical " attached to them. It is also a created universe and St. Augustine's historic saying is quoted, " non in tempore, sed cum tempore, finxit Deus mundum ". And as to the Creator, the facts suggest that we should think of Him as a pure mathematician.

We have come a long way from the days when it was regarded as something like a *tour de force* to be up to date in science and at the same time to believe in a Creator. The theologian who inherits the idealistic tradition to which Dr. Rashdall gave such effective expression in the article with which the volume entitled *Contentio Veritatis* began will welcome the reinforcement from the side of modern science which Sir James provides when he finds science pointing to the conclusion that the objectivity of objects arises from their " subsisting in the mind of some Eternal Spirit " : Sir James makes use of words of Bishop Berkeley.

Yet there are certain difficulties. The idea of a creation in time is much harder to conceive when it is mathematical formulae which give us the final truth (as final as we can have) as to the nature of the universe. What is the meaning of an act of thought which creates a system (the universe) of mathematical relations? It is extremely difficult to ally the idea of creation with that of a series of mathematical abstractions, such as ether and energy have become in these pages. As I understand Sir James, he does not think it likely that the mathematical interpretation of the physical world will be overthrown; at the same time he contemplates the death of the universe when entropy has increased to the furthest possible point. But in what sense can death be predicated of a universe of which we can express no more ultimate truth than that it came into existence through the thought of its Creator, is most accurately described in terms of mathematical formulae, and, presumably, still exists in the thought of its Creator? Such a phrase as " the death of the universe " is intelligible when we are thinking of the universe in terms of physics, but when physics loses itself in mathematics the above words lose their appropriateness.

Further, one may ask how much is involved in the notion that the universe reveals the Creator as a pure mathematician. Clearly, if our universe is what it is because of its creation " cum tempore " through an act of thought, there is nothing absolute about its character. The Creator might have thought it quite differently. And allowing that " death " may come to this universe, what is there to stop the Creator from making another universe which will not reveal Him as a pure mathematician? One need not suppose that His nature is exhausted in what the present order manifests of Him. And as to the present order, this is just one of the connexions in which it is important to remember that that order must not be restricted to its physical setting. There is a great

deal in the order of the universe which may have as much light to throw on the nature of God as anything which can be learnt from a study of the space-time continuum. It is, indeed, only when the universe is examined from one particular view-point that the Creator appears as a mathematician and only a mathematician.

"Mind", says Sir James, "no longer appears as an accidental intruder into the realm of matter; we are beginning to suspect that we ought rather to hail it as the creator and governor of the realm of matter". That is what the Christian theologian has always believed to be the truth as to the relation of God to the world. But I doubt whether he would have said it with any conviction unless he had also felt able to say a number of things which Sir James Jeans does not say, whatever view he may take of them. The theologian approached the problem with a background less "scientific" in the narrower sense of the term, but more inclusive and, therefore, more adequate to the greatness of the theme. If we are going to think as anthropomorphically as Sir James quite rightly does when he is expounding *ex parte Dei* the notion of creation, it will, I believe, be found impossible to abide just in the position where Sir James Jeans leaves us. God will not cease to be thought of as the celestial mathematician; the significance of the validity of mathematical formulae in connexion with the interpretation of the physical order will be fully recognized; but in such conceptions one stage of truth and not its fullness will be confessed.

J. K. MOZLEY.

Grundlagen der Geometrie. By D. HILBERT. Seventh edition. Pp. vii + 326. Rm. 18. 1930. (Teubner.)

Here at last we have a really revised edition of the *Grundlagen*, for the fifth and sixth editions were reprints of the fourth, which itself differed little from the third, published in 1909; and it is likely that this edition, revised as it is in detail, is definitive.

The differences between this and earlier editions begin with the first axiom, and as most of the others have been modified, and as many teachers of Geometry are interested in the assumptions concealed from beginners, it may be useful to set out Hilbert's final formulation.

He divides his axioms, as before, into five groups.

The undefined ideas in Group I (the axioms of connection) are "point", "line", and the notion of "lying on" in such phrases as, the point A lies on the line a, the line b lies on the plane α. The axioms, translated freely, are as follows:

I. 1. Given two points A and B, there is always a line on which both A and B lie.

I. 2. Given two points A and B, there is not more than one line on which both A and B lie.

I. 3. On a line lie always at least two points. There are at least three points which do not lie on the same line.

I. 4. If A, B, C are points not lying on the same line, there is always a plane α on which A, B, C all lie. On each plane lies at least one point.

I. 5. If A, B, C are points not lying on the same line, there is not more than one plane on which they all lie.

I. 6. If two points A, B lie on a line a and on a plane α, then every point which lies on a lies on α.

I. 7. If two planes α, β have a point A lying on both, there is always at least one other point lying on both.

I. 8. There exist at least four points which do not lie on the same plane.

The undefined idea in Group II (the axioms of order) is a relation which relates three points, and is called "between".

II. 1. If a point B is between a point A and a point C, then A, B, C are distinct points lying on the same line, and B is between C and A.

II. 2. If A, C are given points, there is at least one point B which lies on the line AC so that C is between A and B.

II. 3. Of any three points which lie on a line, there is not more than one which is between the other two.

It is now possible to define the new terms used in the next axiom.

II. 4. If A, B, C be points not lying on the same line, and a be a line lying on the plane ABC, but none of A, B, C lie on a, then if the line a cuts the side AB of the triangle ABC, it cuts either the side AC or the side BC.

All the other intuitive theorems on order can now be shown; in particular, we can define the part of a plane "on one side" of a line, and the ideas of ray, angle, the inside of an angle, and so on.

The undefined ideas in Group III are the congruence of point-pairs and the congruence of angles.

III. 1. If A, B are two points lying on a line a, and A' be a point lying on the same or another line a', then on a given ray of the line a' from A', there is always a point B' such that the point-pairs AB and $A'B'$ are congruent.

This relation is denoted by $AB \equiv A'B'$ or $AB \equiv B'A'$ or $BA \equiv A'B'$ or $BA \equiv B'A'$.

III. 2. If $A'B' \equiv AB$ and $A''B'' \equiv AB$ then $A'B' \equiv A''B''$.

III. 3. If A, B, C are three points lying on the same line a, and B is between A and C, and if A', B', C' be three points lying on the same line a', and B' be between A' and C', and if $AB \equiv A'B'$ and $BC \equiv B'C'$, then $AC \equiv A'C'$.

[I must insert a note here. Hilbert, as in earlier editions, defines a "Strecke" AB as a pair of points A, B (in either order). Points between A and B are called points "of" the Strecke AB or points "inside" the Strecke AB. He speaks in III 3 of the common points of two Strecken, meaning points which are points "of" both Strecken. I have altered his statement of III 3, but not, I hope, its meaning, because a straightforward translation might be misleading.]

III. 4. Let $\angle(h, k)$ be an angle in a plane α, and let a line a' be given in a plane α' and also a fixed side of a' in α'. Let h' be a ray of the line a' from a point O'; then in α' there is one and only one ray k' such that $\angle(h, k)$ is congruent to $\angle(h', k')$, and such that all inside points of $\angle(h', k')$ lie on the given side of a'. Each angle is congruent to itself.

III. 5. If for triangles ABC, $A'B'C'$ we have $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle BAC \equiv \angle B'A'C'$, then $\triangle ABC \equiv \triangle A'B'C'$.

Group IV consists of one axiom, the parallel axiom.

Group V consists of two axioms: V 1 that of Archimedes; V 2 the axiom of completeness, much weakened in this edition.

It will be noted that in the axioms of order, it is not now assumed that if A, B are two points, then there is a point between them; in III 1 it is not assumed that the point B' is unique; and the transitive property of the congruence of angles is not assumed. All these properties can be proved from the axioms given.

The simplification of the system given in earlier editions is in part due to Rosenthal, and I should like to mention a minute logical point which may be of some interest. Rosenthal, working on a previous system of Hilbert's, used an axiom, "three points, not lying on the same line, determine a plane", in the following manner. He said: this axiom would be meaningless if there were not at least *one* case in which it was realisable, *i.e.* if there were never three non-collinear points. Hence there do exist three non-collinear points.—This reasoning has been condemned and, I think, wisely. In arguments from axioms, we must work inside the scheme provided by them, and not view them from outside that scheme, as we should do if we allowed ourselves to speak of them as meaningless. Evidently Hilbert is also of that opinion, for in axiom I 3 he *asserts* the existence of three non-collinear points. We may, of course, on another occasion, view the axioms from outside, but what we gain then we may not use in working inside the scheme.

The present formulation of the axioms is a great advance on the older; in earlier editions it seemed that Hilbert had little patience with the foundation work and reserved his interest for the later things, and this was surprising in view of his work on logic in which even more minute details were considered. The result was that in this respect his work was inferior logically to the contemporary work in Italy and the later work in America. This impatience sometimes led to actually incorrect statements, as *e.g.* in theorem 14 of the third edition (numbered 15 in subsequent editions). All this has now been

replaced by careful formulation; for instance in showing that each interval can be bisected, he takes the trouble to prove that its mid-point lies inside the interval.

Is the above formulation of the axioms the best? Such a question is impossible to answer dogmatically because questions of taste are involved. At first sight it would seem easy to say which of two axiomatic foundations of a mathematical doctrine was the better. Count, we might say, the number of undefined ideas, and count the number of axioms, and give the preference to the system involving the smaller number of both. But the matter is not quite so simple as this, for two axioms might be replaced by one, by simply altering the phrasing or the symbolism; also axioms differ in strength, one may include another. And there is a more serious difficulty than these. Let us take a non-geometric instance: a "field" is a set of elements which can be added and multiplied together, the laws of addition and multiplication being those of ordinary algebra; and it is easy to set out the axioms of a field, using as undefined ideas those of addition and multiplication, and taking as axioms the formal laws of addition and multiplication. But it is also possible to use only *one* undefined idea, from which both addition and multiplication can be derived, and to set out axioms, fewer in number than before, which can serve as a basis of the whole theory. But by so doing, we should make the theory much more complicated, and, what is more serious, we should disguise its true nature.

Turning now to geometry, it is possible to replace Hilbert's axioms of Groups I and II by a much smaller number, and to use instead of his undefined terms only two, namely "point" and "between". Line, plane, can be defined in terms of these. And it is possible to use as the undefined idea in the theory of congruence, the single idea of the congruence of point-pairs, and to derive the notion of congruent angles from it.

It is the reviewer's opinion that these reductions do not, in this instance, materially complicate matters, and that they are natural and in accordance with the spirit of geometry. This is, of course, not intended as an adverse criticism of Hilbert's formulation, which would be merely an impertinence on my part. Hilbert's axiom-groups are now universally known, they could not be scrapped without changing the nature of the book, and if they are not the minimum set, they are now at any rate in a satisfactory form. Moreover, the minimum set is due to work which owed much of its impulsion to the *Grundlagen* itself.

We now consider the deductions from the axioms and deal as before mainly with the differences from earlier editions. Proofs are now given of some of the early theorems on order, but it is still stated that it can be shown "ohne erhebliche Schweirigkeit" that a simple plane polygon separates its plane into two parts; the whole theory of congruence is revised; in Chapter II is proved under the assumptions of the axioms of Groups I, II, III and V 1, the theorem of Legendre that the sum of the three angles of a triangle cannot be greater than two right angles, and, from I, II, III, that if the sum is two right angles in one triangle, it is so in all (note that parallels may not be used in these two theorems); the theory of the areas of polygons is handled somewhat differently from before, and fuller consideration is given of the consequences of denying the axiom of Archimedes; in Chapter V the proof that Desargues' theorem on perspective triangles cannot be shown from the planar axioms I 1-3, II, III 1-4, IV, V (III 5 is omitted) is made to depend on Moulton's non-Desarguesian geometry instead of on the one previously used; the subsequent introduction of an analytical geometry without the use of the congruence axioms is also much simplified, and in the chapter on Pascal's theorem (=Pappus' theorem), the key-stone theorem of Hessenberg is added; the work on gauge constructions in Chapter VII has been improved by the use of recent algebraic results of Artin. A large number of other changes, small and big, occur throughout, and if nothing that is actually new has been added, the additions are most of them due to investigations suggested by previous editions.

Of the 323 pages of this edition, 198 consist of appendices. Of these numbers 1, 3, 4, 6, 7 are unchanged or changed but little. They deal with (1) geometries in which the shortest join of two points is the straight join, (3) hyper-

bolic geometry, (4) a marvellous piece of reasoning in which plane geometry is built up from three axioms of motion in a continuous manifold, (6), (7) the foundations of arithmetic.

Appendix 2 deals with those surprising geometries in which the angles at the base of an isosceles triangle are *not* equal. This appendix has been completely recast in the light of Rosemann's dissertation. Appendix 5 shows that there is no regular analytic surface of constant negative curvature which is free from singularities in the finite part of space, and that the only closed surface of constant positive curvature, free from singularities, is the sphere. (Thus the simplest surface of constant negative curvature is the pseudo-sphere, generated by revolving a tractory about its directrix). The geometric proof, in earlier editions, of the first theorem stated, was attacked by Liebmann as incomplete and defended by Bieberbach. It was undeniably difficult and too concisely phrased in places, and has now been replaced by an analytical proof due to Holmgren and Blaschke, but I regret the omission of the beautiful earlier proof.

Finally, there are three additional appendices (pp. 262-323) consisting of lectures by Hilbert on his metamathematics. (Über das unendliche, *Math. Ann.* 95. Die Grundlagen der Mathematik, *Hamburg Abh.* 6. Probleme der Grundlegung der Mathematik, *Math. Ann.* 102.) There is a certain amount of repetition in these, but in reading them one after the other, one does get a very clear idea of Hilbert's method in logic. This has been described in the *Gazette* (Dec. 1928, p. 273) in a review of a book by Hilbert and Ackermann, and it is unquestionably a very considerable advance beyond Whitehead and Russell's *Principia*. Hilbert is of opinion that his method should make the foundations of mathematics secure for all time, but it is at present in programme form, though it is being rapidly worked out.

It is a matter of regret that the new edition of this great classic—worthy to be ranked with Euclid's *Elements*—should cost so very much more than the earlier editions. Can nothing be done to stem the tide of rising prices at home and abroad ?

H. G. F.

Die anschauliche Natur der geometrischen Grundbegriffe. By R. Potthof. Pp. 28. Rm. 1. 1930. (Aschendorffsche Verlagsbuchhandlung, Münster in Westfalen.)

This pamphlet is concerned with the connection between abstract geometry and experience, and attempts to justify the use of that geometry in experience. We must distinguish between the following questions : How does, as a matter of fact, a human infant arrive at geometric notions ? How did our ancestors, human and pre-human, acquire these notions ? How would a highly intelligent being, with a body like ours, if suddenly introduced into the world, acquire these notions ?

The author deals essentially with the last question, and assumes that the being tackles the problem as an engineer might. Beginning with the ideas of the contact of bodies and of moving bodies, derived from observation, he introduces the sphere as a body whose surface can move over itself in all ways. Planes are introduced as follows : if we have three bodies such that certain parts a, b, c of their surfaces can be brought into contact with each other (a with b, b with c, c with a), then these parts are portions of planes. Considering the motions which preserve contact when one sphere moves inside an equal hollow sphere, and when one plane slides along another, he gets the ideas of rotation round a point and of motions conserving a plane. If two spheres be connected by a rigid bar, it is still possible to move them about themselves ; hence arises the notion of rotation round an axis. These remarks will suffice to show the method of the tract, but much more detail would be needed to complete the bridge between experience and geometry, for there is a danger of using geometric concepts and results in the course of the experiments from which they are supposed to be derived, and this pamphlet is so short that the author has not succeeded in making clear that he has avoided this error.

H. G. F.

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